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Remarks on Effect Algebras

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Abstract Erik M. Alfsen and Frederic W. Shultz had recently developed the characterisation of state spaces of operator algebras. It established full equivalence (in the mathematical sense) between the Heisenberg and the Schrödinger picture, i.e. given a physical system we are able to construct its state space out of its observables as well as to construct algebra of observables from its state space. As an underlying mathematical structure they used the theory of duality of ordered linear spaces and obtained results are valid for various types of operator algebras (namely C^*, von Neumann, JB and JBW algebras). Here, we show that the language they developed also admits a representation of an effect algebra.

Keywords Effect algebras · State spaces · Operator algebras

1 Introduction

Quantum Mechanics emerged from two different ideas: the Heisenberg matrix mechanics and the Schrödinger wave mechanics [4, 5, 11]. It was quickly recognised that the Heisenberg's approach leads to a Jordan algebra[1] structure [9] while the Schrödinger's mechanics relies on the time evolution of states. The question whether these two approaches are equivalent was posed. Although expectation values of observables were found to be the same in both formulations and the time evolutions are compatible it does not mean that they are

[1] Recall that a Jordan algebra is a vector space over a scalar field equipped with a commutative bilinear product $\circ$, such that $(a^2 \circ b) \circ a = a^2 \circ (b \circ a)$. Note, that this product, in general, is not assumed to be associative.

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equivalent in the mathematical sense. In particular, in the Heisenberg formulation we are able to construct the state space of a physical system from its algebra of observables. The opposite programme, i.e. constructing the algebra of observables form the state space, needs the characterisation of state spaces of operator algebras among all other convex sets.

The final answer to this question was recently given by Alfsen and Shultz in [1, 2]. They developed an axiomatic characterisation of those convex sets that are state spaces of C^*-algebras, von Neumann algebras and some general classes of Jordan algebras. As an underlying abstract mathematical structure they used the theory of duality of ordered linear spaces.

On the other hand, in the so called operational quantum physics a special emphasise is put on the occurrence of particular outcomes of measurements—effects [3]. This leads to the theory of effect algebras.

The aim of my talk is to bring together these two approaches. More precisely it will be shown that the Alfsen and Shultz programme admits a concrete representation of an effect algebra.

2 Basic Definitions

In the most typical example of the Heisenberg approach it is said that observables are represented by self-adjoint elements of a von Neumann algebra $\mathfrak{M}$. Then normal states are normalised, linear, positive functionals from its predual $\mathfrak{M}_*$ (i.e. $(\mathfrak{M}_*)^* \cong \mathfrak{M}$). It is a well known fact that all normal states can be represented by density matrices. Given a state $\omega \in \mathfrak{M}_*$ and an observable $a \in \mathfrak{M}$ a value $\langle a, \omega \rangle = \omega(a)$ corresponds to the mean value of results of measuring observable a. Note that the functional analysis gives not merely mean value but actually the whole probability distribution of the random variable corresponding to the observable a.

In this strategy, propositions, i.e. yes-no measurements, are represented by orthogonal projections in $\mathfrak{M}$. We can imagine such measurement as sending the beam of particles in the state ω through the device that splits them into two beams, in states ω_1, ω_0, corresponding to measured results "yes" and "no". If $p \in \mathfrak{M}$ is the projection representing such device, then we can write states ω_1, ω_0 as a transformed state ω, explicitly: $\omega_1 = U_p^* \omega$, $\omega_0 = U_{p'}^* \omega$, where $U_p a = pap$ (dual to U_p^*), and $p' = 1 - p$.

This example yields some hints for a generalisation of the Heisenberg scheme [2] but before we present it, we need to recall some definitions.

Firstly let us recall that the ordered linear space is a pair $(\mathcal{A}, \mathcal{A}^+)$ (in the sequel, for simplicity, we will consider only real linear spaces) , where $\mathcal{A}^+ \subset \mathcal{A}$ is a *proper positive cone*, i.e. (i) $x, y \in \mathcal{A}^+ \implies x + y \in \mathcal{A}^+$, (ii) for a positive scalar λ, $x \in \mathcal{A}^+ \implies \lambda x \in \mathcal{A}^+$ and (iii) $\mathcal{A}^+ \cap (-\mathcal{A}^+) = 0$. Then the partial order relation is defined as follows: $x \le y \iff y - x \in \mathcal{A}^+$. Elements of $\mathcal{A}^+$ are called *positive*. A linear space $\mathcal{A}$ is said to be *generated* by a positive cone $\mathcal{A}^+$ iff $\mathrm{co}(\mathcal{A}^+ \cup (-\mathcal{A}^+)) = \mathcal{A}$.

Of course the positive cone is a convex set. We recall that *the face* $\mathcal{F}$ in a convex set $\mathcal{C}$ (in some linear space $\mathcal{A}$) is its subset such that the following condition holds: $x, y \in \mathcal{C}$, $(1-\lambda)x + \lambda y \in \mathcal{F} \implies x, y \in \mathcal{F}$, for $\lambda \in [0, 1]$. In particular, extreme points of a convex set are one-point faces. Thus, face generalises notion of extreme points. We say that the face is *exposed* when there exists one closed supporting hyperplane H of $\mathcal{C}$ (in $\mathcal{A}$) such that $\mathcal{F} = \mathcal{C} \cap H$ (for more details see [1]).

In the sequel we will be interested in following two types of linear spaces (see [2]):

Definition 1 (Order unit space) An ordered normed linear space $\mathfrak{A}$ over $\mathbb{R}$ with a closed positive cone and an element 1 satisfying:

$$\|a\| = \inf\{\lambda > 0; -\lambda 1 \leq a \leq \lambda 1\} \tag{1}$$

is called an *order unit space.*

Note that order unit space is generated by its positive cone.

Definition 2 (Base norm space) Let $\mathfrak{B}$ be an ordered normed linear space with a generating cone $\mathfrak{B}^+$. If $\mathfrak{B}^+$ has a base K located on a hyperplane H $(0 \notin H)$ such that $\mathrm{co}(K \cup -K)$ is the closed unit ball $\mathfrak{B}_1$ of $\mathfrak{B}$ then $\mathfrak{B}$ is called the *base norm space* and K is called the *distinguished base of* $\mathfrak{B}$.

Because the theory of duality plays a crucial rôle in these considerations, for sake of completeness we should recall the definition of a *dual pair.* Let $\mathfrak{A}$, $\mathfrak{B}$ are linear spaces over a scalar field $\mathcal{S}$, and $\langle\cdot,\cdot\rangle\colon \mathfrak{A} \times \mathfrak{B} \to \mathcal{S}$ is a bilinear form such that:

$$\forall x \in \mathfrak{A} \setminus \{0\}, \exists y \in \mathfrak{B}: \quad \langle x, y\rangle \neq 0$$
$$\forall y \in \mathfrak{B} \setminus \{0\}, \exists x \in \mathfrak{A}: \quad \langle x, y\rangle \neq 0.$$

A triple $(\mathfrak{A}, \mathfrak{B}, \langle\cdot,\cdot\rangle)$ is called a *dual pair.*

It is an interesting fact, that a dual to a base norm space is an order unit space and a dual to an order unit space is a base norm space ([1], Theorem 1.19).

Definition 3 An order unit space $\mathfrak{A}$ and a base norm space $\mathfrak{B}$ are in *separating order and norm duality* under the bilinear form $\langle\cdot,\cdot\rangle$ when they are in separating duality and for $a \in \mathfrak{A}$, $\rho \in \mathfrak{B}$:

$$a \geq 0 \iff \langle a, \rho\rangle \geq 0 \quad \forall \rho \geq 0 \tag{2}$$

$$\rho \geq 0 \iff \langle a, \rho\rangle \geq 0 \quad \forall a \geq 0 \tag{3}$$

$$\|a\| = \sup_{\|\sigma\|\leq 1} |\langle a, \sigma\rangle| \tag{4}$$

$$\|\rho\| = \sup_{\|b\|\leq 1} |\langle b, \rho\rangle| \tag{5}$$

From now on $\mathfrak{A}$, $\mathfrak{B}$ will always denote a pair of an order unit space and base norm space in separating order and norm duality.

Now we should introduce the candidate for a generalised proposition. Note that in the following definition we are describing the concept of a projection in the more general framework (again for details see [2]).

Definition 4 A projection is a linear map $P\colon \mathfrak{A} \to \mathfrak{A}$ $(P\colon \mathfrak{B} \to \mathfrak{B})$ satisfying $P^2 = P$. A projection P is called *positive* iff $P(\mathfrak{A}^+) \subseteq \mathfrak{A}^+$ $(P(\mathfrak{B}^+) \subseteq \mathfrak{B}^+)$. P is normalised iff $P1 \leq 1$.

Definition 5 Two projections P, Q are said to be *complementary* if

$$\ker^+ P = \mathrm{im}^+ Q, \tag{6}$$

$$\text{im}^+ P = \ker^+ Q, \tag{7}$$

where $\ker^+ P = \ker P \cap \mathfrak{A}^+$, $\text{im}^+ P = \text{im} P \cap \mathfrak{A}^+$. P is said to be *bicomplemented* if there exists weakly-continuous positive projection Q on $\mathfrak{A}$ such that P, Q are complementary and P^*, Q^* are complementary (where P^* denotes dual projection).

Definition 6 A bicomplemented normalised weakly-continuous positive projection P on $\mathfrak{A}$ is called a *compression*. The element $p \in \mathfrak{A}_1^+$, $p = P1$ is called its *associated projective unit*, and the face $\mathcal{F}_P = K \cap \text{im} P^*$ is called its *associated projective face*.

There is the one-to-one correspondence between compressions and projective units and projective faces. Moreover, if p is an associated projective unit of P, then the associated projective unit of its complement P', is equal to $p' = 1 - p$ (see [2]).

Definition 7 Two compressions P, Q are said to be *orthogonal* ($P \perp Q$) iff $PQ = 0$. We also write $p \perp q$.

Proposition 1 [2, 7.34] *The following conditions are equivalent*:

(i) $QP = 0$,
(ii) $p \leq q'$,
(iii) $PQ = 0$.

If we take, for example, $\mathfrak{A} = \mathfrak{M}$, $\mathfrak{B} = \mathfrak{M}_*$, then all compressions on $\mathfrak{A}$ have the form $U_p = pap$, where p is ordinary defined orthogonal projection from $\mathfrak{M}$ [2].

From now on (following [2]) we make the additional assumption that $\mathfrak{A}, \mathfrak{B}$ is a such pair that each exposed face of the distinguished base K of $\mathfrak{B}$ is projective. We will say that such $\mathfrak{A}, \mathfrak{B}$ satisfy our *basic hypothesis*. In particular it is true when $\mathfrak{A}$ is the self-adjoined part of a von Neumann algebra and $\mathfrak{B}$ is self-adjoined part of its predual or $\mathfrak{A}$ is a JBW-algebra[2] and $\mathfrak{B}$ is its predual [2].

Now we are prepared to proceed with a generalisation of the mentioned example of the Heisenberg approach. This can be done by replacing the pair of von Neumann algebra and its predual by a pair of a order unit space $\mathfrak{A}$ and a base norm space $\mathfrak{B}$ [2], such that $\mathfrak{A} = \mathfrak{B}^*$. Then normal states are represented by the distinguished base $K \subset \mathfrak{B}$ (which is a convex set) and it is argued [2] that propositions are represented by compressions (or equivalently by their associated projective units or faces).

As we are not interested only in von Neumann algebras, we can say that given a pair of order unit space and base norm space $\mathfrak{A}, \mathfrak{B}$, satisfying the basic hypothesis, elements of $\mathfrak{A}$ represent observables, and elements of distinguished base of $\mathfrak{B}$ represent states. Then propositions are being represented by compressions [2].

The advantage of using the described generalisation lies in the fact that it allows complete characterisation of state spaces of von Neumann algebras, JB, JBW and C^*-algebras. Namely, in the abstract setting (e.g. as proposed by [10]), the considered scheme provides a criterion for a given convex set which guarantees that this convex set is isomorphic to the

[2] We recall here that JBW-algebra is a JB-algebra that is monotone complete and admits separating set of normal states. JB-algebra is a Jordan algebra, with an identity element and norm satisfying: $\|a \circ b\| \leq \|a\| \|b\|$, $\|a^2\| = \|a\|^2$ and $\|a^2\| \leq \|a^2 + b^2\|$.

state space of some operator algebra. Then, we are able to reconstruct this algebra of observables [2]. All that gives us the full equivalence between Schrödinger and Heisenberg picture, not only in sense of a mean value. We emphasise that for a given algebra of observables of a certain physical system we can construct its state space, as well as for a given state space we can construct its algebra of observables.

3 Compressions and the Effect Algebra

As mentioned in the introduction the generalised scheme gives a realisation of the effect algebra, what is to be proved now. Firstly recall the definition of an effect algebra. Consider a tuple $(E, 0, 1, \oplus)$, where $\oplus$ is a partially defined binary relation and 0, 1 are distinct elements of E. If $a \oplus b$ is defined then we write $a \perp b$. If following conditions hold:

(i) $a \perp b \implies b \perp a$ and $a \oplus b = b \oplus a$,
(ii) $a \perp b$ and $c \perp (a \oplus b) \implies b \perp c$ and $a \perp (b \oplus c)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,
(iii) $\forall a \in E, \exists a' \in E : a \perp a'$ and $a \oplus a' = 1$,
(iv) $a \perp 1 \implies a = 0$,

we say that $(E, 0, 1, \oplus)$ is an effect algebra.

Proposition 2 [2, 8.1 and 8.10] *Let $\mathfrak{A}, \mathfrak{B}$ satisfy our basic hypothesis. Moreover let $\mathcal{C}$ denote the set of all compressions on $\mathfrak{A}$, $\mathcal{P}$ the set of all projective units and $\mathcal{F}$ the set of all projective faces. Then $\mathcal{C}, \mathcal{P}, \mathcal{F}$ are all orthomodular isomorphic lattices with $\vee, \wedge$ defined by*:

$$R = P \vee Q \quad \iff \quad \mathcal{F}_R = (\mathcal{F}'_P \cap \mathcal{F}'_Q)' \tag{8}$$

$$S = P \wedge Q \quad \iff \quad \mathcal{F}_S = \mathcal{F}_P \cap \mathcal{F}_Q \tag{9}$$

Lemma 1 [2, 8.6] *Let $\mathfrak{A}, \mathfrak{B}$ satisfy our basic hypothesis. If P, Q, R are mutually orthogonal compressions, then $P \perp (Q \vee R)$.*

Proposition 3 [2, 8.8] *Let $\mathfrak{A}, \mathfrak{B}$ satisfy our basic hypothesis. If $p_1, \ldots, p_n$ are projective units, then following are equivalent*:

(i) $\sum_{i=1}^n p_i \le 1$,
(ii) $p_i \perp p_j, \quad i \neq j$,
(iii) $\bigvee_{i=1}^n p_i = \sum_{i=1}^n p_i$.

Lemma 2 *Let $\mathfrak{A}, \mathfrak{B}$ satisfy our basic hypothesis. If $P \perp Q$ and $(P \vee Q) \perp R$ then P, Q, R are mutually orthogonal.*

Proof If $P \perp Q$, then $PQ = 0$ and by Proposition 1 $p \le q' = 1 - q$, where $p = P1, q = Q1$. By the same arguments $(P \vee Q)1 \le 1 - r$. But as lattices $\mathcal{C}$ and $\mathcal{P}$ are isomorphic, $(P \vee Q)1 = p \vee q$. Furthermore, as $p + q \le 1$ Proposition 3(iii) implies $p \vee q = p + q$. Consequently $p + q + r \le 1$, what means that P, Q, R are mutually orthogonal. □

Theorem 1 *Let $\mathfrak{A}, \mathfrak{B}$ satisfy our basic hypothesis. Define $P \oplus Q = P \vee Q$ whenever $P \perp Q$ and let $\mathbf{0}$ denotes the null compression while $\mathbf{1}$ denotes the compression onto $\mathfrak{A}$. Then $(\mathcal{C}, \oplus, \mathbf{0}, \mathbf{1})$ is a realisation of the effect algebra.*

Proof By the Lemma 1, if $P \perp Q$, then $Q \perp P$, so $P \oplus Q$ and $Q \oplus P$ are both well defined and are obviously equal.

If $P \oplus Q$ and $(P \oplus Q) \oplus R$ are defined, it means that $P \perp Q$ and $(P \vee Q) \perp R$. So these are mutually orthogonal (by Lemma 2). Thus $Q \perp R$ and $P \perp (Q \vee R)$ hence $Q \oplus R$ and $P \oplus (Q \oplus R)$ are also defined. Equality $(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$ is obvious for a lattice.

For each $P \in \mathcal{C}$ its complement is uniquely defined. As the consequence of complementarity, $PP' = \mathbf{0}$, so $P \oplus P'$ is defined and equal to $\mathbf{1}$ by the orthomodularity of the lattice $\mathcal{C}$.

Eventually, if $P \oplus \mathbf{1}$ is defined, then $p \leq 1' = 0$, so $p = 0$ and $P = \mathbf{0}$. □

4 "AS" Compressions vs. "F" Compressions

It is not the first time that the word "compression" appears in the context of effect algebras so we feel obliged to clarify this point. David J. Foulis [6] introduced compressions on partially ordered abelian groups and the concept of compression bases in unital groups [7] (further referred as "F" compressions). These are not exactly the same as those defined in Sect. 2. To distinguish them, compressions defined in Definition 6 will be denoted by "AS" compressions.

"F" compressions were adopted on the ground of effect algebras by Stan Gudder [8]. Moreover it is easily seen that "F" compressions are defined on more general structure than "AS" compressions. However we will show that "AS" compressions are "F" compressions if we restrict ourselves to group properties of a ordered linear space. To this end, firstly we recall the definition of "F" compressions.

Definition 8 [6, 2.1 and 2.4] Let G be a partially ordered abelian group with order unit 1. A mapping $P: G \to G$ is called a *retraction* on G iff:

(i) P is an order-preserving group endomorphism,
(ii) $P1 \leq 1$,
(iii) $0 \leq a \leq P1 \implies Pa = a$,
(iv) P is idempotent.

The retraction P with *focus* $p = P1$ is a *compression* (*"F" compression*) iff $\forall a \in G$, $0 \leq a \leq 1$, $Pa = 0 \implies a \leq 1 - p$.

Lemma 3 *Assume that P is an "AS" compression. If $a \in \operatorname{im}^+ P$, then $Pa = a$.*

Proof If $a \in \operatorname{im}^+ P$ then there exists $b \in \mathfrak{A}^+$ such that $Pb = a$. Then $Pa = P(Pb) = Pb = a$. □

Proposition 4 *Let P be an "AS" compression. Then P is a "F" compression.*

Proof $\mathfrak{A}$ is an order unit space, so $(\mathfrak{A}, +)$ is a partially ordered abelian group with order unit.

Obviously conditions (ii) and (iv) of the definition are satisfied. To show (i) let $a \leq b$. Then $b - a \geq 0$, so $P(b - a) \geq 0$ (positivity). Hence $Pa \leq Pb$. (iii) If $0 \leq a \leq P1$, then, by Proposition 7.31 in [2], $a \in \mathfrak{A}_1 \cap \operatorname{im} P$. Therefore $a \in \operatorname{im}^+ P = \operatorname{im} P \cap \mathfrak{A}^+$. Thus $Pa = a$. Consequently P is a retraction.

Now, let $c \in \mathfrak{A}_1^+$. If $Pc = 0$ then $c \in \ker^+ P = \mathrm{im}^+ P'$ and by Proposition 7.31 in [2] $c \in [0, p'] = [0, 1 - p]$. □

"AS" compressions and "F" compressions are various generalisations of maps on C^*-algebras of the form $a \mapsto pap$. Proposition 4 suggests that "F" compressions are more general than "AS" compressions. However we must remember that "AS" compressions are defined in a framework of a dual pair which possesses a richer structure than a group. We emphasise that the duality has an important feature. It allows to interpret "AS" compressions as a generalised propositions. Moreover, note that Definition 6 as well as comments following that definition establish the relations between compressions and geometric ingredients of state space characterisation. This indicates that the dual pair strategy seems to be the best adjusted to Quantum Mechanics.

Finally note that [8] introduced "F" compressions on effect algebras, while here we presented a representation of an effect algebra in terms of "AS" compressions.

5 Conclusions

To sum up, there exists a mathematical structure that allows one to treat Heisenberg's and Schrödinger's approach on equal footing [1, 2] and it provides general description of observables, states and a relation between them.

The concept of compressions is of fundamental importance for characterisation of state spaces of operator algebras (namely C^*, W^*, JB and JBW algebras) among all other convex sets.

The same notion provides a new representation of an effect algebra. This representation gains in interest if we realise that it contains the most common (at least for physicists) representation of an effect algebra, namely $B(\mathcal{H})$[3], as the special case.

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[3] Bounded, linear operators acting on Hilbert space $\mathcal{H}$.